II Introduction to Topology

Everyone should be familiar with continuous functions and convergence in metric spaces (from Analysis I). We discuss the most general context in which one can consider these ideas

A. Topological Spaces

let X be any set

a collection of subsets $\mathcal{T}$ of X is a topology for X if

empty set

1) $\varnothing$ and X are in $\mathcal{T}$
2) A and B in $\mathcal{T} \Rightarrow (A \cap B)$ in $\mathcal{T}$
3) if $\{A_\alpha\}_{\alpha \in J}$ is a collection of sets in $\mathcal{T}$

   then $\bigcup_{\alpha \in J} A_\alpha$ is in $\mathcal{T}$

here J is an index set

e.g. $J = \{1, 2\}$, then $\{A_\alpha\}_{\alpha \in J}$ means $\{A_1, A_2\}$

$J = \mathbb{Z}$, then $\{A_\alpha\}_{\alpha \in J}$ means $\{\dots A_{-2}, A_{-1}, A_0, \dots\}$

note: property 2) $\Rightarrow$ any finite intersection of sets in $\mathcal{T}$ is in $\mathcal{T}$

$$\text{e.g. } A \cap B \cap C = \underbrace{\underbrace{(A \cap B)}_{\text{in } \mathcal{T}} \cap C}_{\text{in } \mathcal{T}}$$

a topological space is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X

elemets of $\mathcal{T}$ are called open sets

examples: $X = \{a, b, c\}$

$\mathcal{T}_1 = \{X, \varnothing, \{a\}, \{b\}, \{a, c\}\}$

is not a topology on X

$\mathcal{T}_2 = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\}$

is not a topology on X

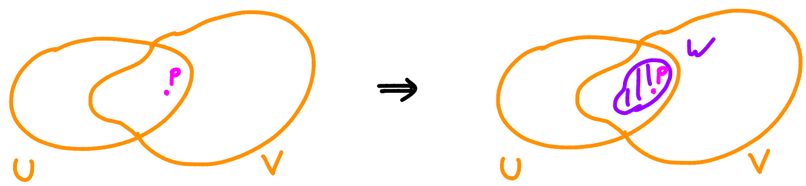
$\mathcal{T}_3 = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$

is a topology on X

to describe interesting topologies we need a new idea

a collection $\mathcal{B}$ of subsets of X is called a basis for a topology on X if

1) X is a union of sets in $\mathcal{B}$
2) if $U, V \in \mathcal{B}$ and $p \in U \cap V$,
then $\exists W \in \mathcal{B}$ s.t. $p \in W \subset U \cap V$



lemma 1:

given a basis $\mathcal{B}$ for a topology on X
let $\mathcal{T}_{\mathcal{B}} = \{$collection of all unions of sets in $\mathcal{B}\}$
Then $\mathcal{T}_{\mathcal{B}}$ is a topology on X

Proof: need to see $\mathcal{T}_{\mathcal{B}}$ satisfies 1)-3) in def^n of topology

1) X is in $\mathcal{T}_{\mathcal{B}}$ by condition 1) of def^n of $\mathcal{B}$
$\emptyset$ is in $\mathcal{T}_{\mathcal{B}}$ since the "union of no sets in $\mathcal{B}$" is $\emptyset$ (by convention)

2) if A and B are in $\mathcal{T}_{\mathcal{B}}$ then

$$A = \bigcup_{\alpha \in J} U_\alpha \quad \text{and} \quad B = \bigcup_{\beta \in J} V_\beta$$

for $U_\alpha, V_\beta \in \mathcal{B}$

now if $p \in A \cap B$ then there is some $\alpha_0 \in J$, $\beta_0 \in I$
s.t. $p \in U_{\alpha_0}$ and $p \in V_{\beta_0}$ i.e. $p \in U_{\alpha_0} \cap V_{\beta_0}$

thus by condition 2) of $\mathcal{B}$ $\exists W_p \in \mathcal{B}$ such that

$$p \in W_p \subset U_{\alpha_0} \cap V_{\beta_0} \subset A \cap B$$

$$\text{so } A \cap B = \bigcup_{p \in A \cap B} W_p$$

$\supseteq$ is obvious
$\subseteq$ is clear with a moments thought!

3) if $\{A_\alpha\}_{\alpha \in J}$ is a collection in $\mathcal{T}_{\mathcal{B}}$, then

$$A_\alpha = \bigcup_{\alpha \in J_\beta} U^\alpha_\beta \quad \text{for } U^\alpha_\beta \in \mathcal{B}$$

$$\text{so } \bigcup_{\alpha \in J} A_\alpha = \bigcup_{\substack{\alpha \in J \\ \beta \in J_\alpha}} U^\alpha_\beta \quad \text{is in } \mathcal{T}_{\mathcal{B}}$$

$\therefore \mathcal{T}_{\mathcal{B}}$ is a topology on X ▦

examples:

1) let (X,d) be a metric space

recall, this means X is a set and

$$d : X \times X \longrightarrow \mathbb{R}$$

is a function satisfying

1) $d(x,y) \geq 0 \quad \forall\, x,y \in X$
2) $d(x,y) = 0 \iff x = y$
3) $d(x,y) = d(y,x)$
4) $d(x,z) \leq d(x,y) + d(y,z)$

then $\mathcal{B}_d = \{ B_r(x) \text{ for all } r > 0 \text{ and } x \in X\}$ is a basis for a topology on X

here $B_r(x) = \{ y \in X \text{ s.t. } d(x,y) < r \}$

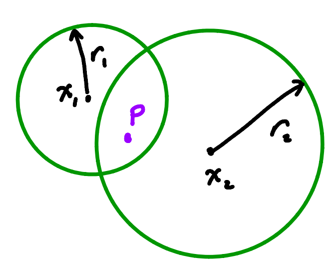
let's check this

1) $X = \bigcup_{x \in X} B_1(x)$ so X is a union of elements in $\mathcal{B}_d$

2) given $B_{r_1}(x_1), B_{r_2}(x_2) \in \mathcal{B}_d$ and a point $p \in B_{r_1}(x_1) \cap B_{r_2}(x_2)$

set $\varepsilon = \min\{ r_1 - d(x_1,p), r_2 - d(x_2,p) \}$

note if $z \in B_\varepsilon(p)$ then



$$\begin{aligned} d(x_1,z) &\leq d(x_1,p) + d(p,z) \\ &< d(x_1,p) + r_1 - d(x_1,p) = r_1 \end{aligned}$$

so $z \in B_{r_1}(x_1)$

similarly $z \in B_{r_2}(x_2)$ so

$$p \in B_\varepsilon(p) \subset B_{r_1}(x_1) \cap B_{r_2}(x_2)$$

thus $\mathcal{B}_d$ is a basis for a topology on X

We call the topology $\mathcal{T}_d$ induced by $\mathcal{B}_d$ the <u>metric topology</u> on X (induced by d)

e.g. $X = \mathbb{R}^n$

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2} \qquad \text{where } x = (x_1, \dots, x_n)$$

is the Euclidean metric on $\mathbb{R}^n$

so d gives $\mathbb{R}^n$ a metric topology

this, of course, is the topology studied in calculus/analysis

also consider

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

and

$$d_2(x,y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$$

<u>exercise</u>: 1) these are metrics on $\mathbb{R}^n$

2) the topologies $\mathcal{T}_d = \mathcal{T}_{d_1} = \mathcal{T}_{d_2}$ are the same!

2) let $(X, \mathcal{T})$ and $(Y, \mathcal{T})$ be two topological spaces

set $\mathcal{B} = \{U \times V : U \in \mathcal{T} \text{ and } V \in \mathcal{T}'\}$

<u>Claim</u>: $\mathcal{B}$ is a basis for a topology on $X \times Y$

indeed: 1) $X \times Y \in \mathcal{B}$ so $X \times Y$ is a union of elts in $\mathcal{B}$

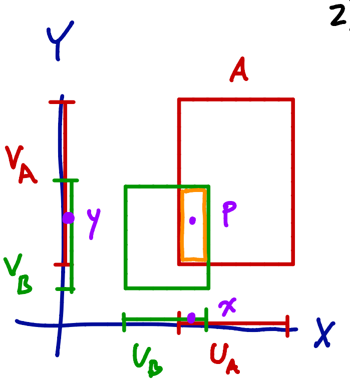
2) if $A, B \in \mathcal{B}$ then $\exists\, U_A, U_B \in \mathcal{T}$ and $V_A, V_B \in \mathcal{T}'$ such that

$$A = U_A \times V_A \quad \text{and} \quad B = U_B \times V_B$$

if $p = (x,y) \in A \cap B$ then $x \in U_A \cap U_B \in \mathcal{T}$

$y \in V_A \cap V_B \in \mathcal{T}'$

so $p \in \overbrace{(U_A \cap U_B) \times (V_A \cap V_B)}^{\in \mathcal{B}} \subset A \cap B$



so $\mathcal{B}$ a basis for a topology on $X \times Y$

the topology on $X \times Y$ induced by $\mathcal{B}$ is called the <u>product topology</u>

<u>exercise:</u> Show that the metric topology on $\mathbb{R}^2$ is the same as the product topology on $\mathbb{R} \times \mathbb{R}$ where $\mathbb{R}$ is given the metric topology

another way to get a topology is as follows

let $(X, \mathcal{T})$ be a topological space

$A \subset X$ a subset

set $\mathcal{T}_A = \{ A \cap U : U \in \mathcal{T} \}$

<u>exercise:</u> $\mathcal{T}_A$ is a topology on A

$\mathcal{T}_A$ is called the <u>subspace topology</u> on A

<u>exercise:</u>

1) If (X, d) a metric space and $A \subset X$ then A has an induced metric d_A

Show $(\mathcal{T}_d)_A = \mathcal{T}_{(d_A)}$

subspace topology of metric topology on X

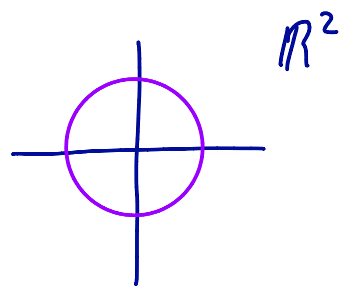
metric topology on A induced by d_A

2) $\mathbb{R}^1 \subset \mathbb{R}^2$ as the x-axis, then the subspace topology on $\mathbb{R}^1$ coming from $\mathbb{R}^2$ with the metric topology is the metric topology on $\mathbb{R}^1$

<u>examples:</u>

1) $S^1 = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$

gets a topology from $\mathbb{R}^2$



more generally

$S^n = \{ (x_0, \dots x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i^2 = 1 \}$

gets a topology from $\mathbb{R}^{n+1}$

2) $\mathbb{Z} \subset \mathbb{R}$ gets a topology from $\mathbb{R}$ (What is it?)

integers

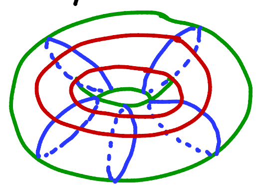
3) $[0,1] \subset \mathbb{R}$ gets a topology from $\mathbb{R}$

note: open sets in $[0,1]$ are unions of

$$[0,b) \qquad 0<a<b<1$$
$$(a,b)$$
$$(a,1]$$
$$[0,1]$$

so open sets in a subspace topology need not be open in original space!

4) subspace topologies + product topologies give a topology on $S^1 \times S^1$



and more generally $S^n \times S^m$

B. Limit Points and Sequences

If A is a subset of a topological space $(X, \mathcal{T})$, then $p \in X$ is a limit point of A if for each open set U containing p we have

$$A \cap (U - \{p\}) \neq \emptyset$$

the closure of A is the set containing A and all the limit points of A, denote the closure by $\bar{A}$

a set C is called closed if it contains all its limit points

lemma 2:

1) $\bar{A}$ is closed (i.e. $\bar{\bar{A}} = \bar{A}$)
2) A is closed $\Leftrightarrow$ $X-A$ is open
3) a finite union of closed sets is closed
4) any intersection of closed sets is closed

Proof:

2) ($\Rightarrow$) if A is closed then any $p \in X-A$ is <u>not</u> a limit pt of A, so $\exists$ some open set U_p such that

$$U_p \cap A = (U_p - \{p\}) \cap A = \emptyset$$

that is $U_p \subset X-A$

so $X-A = \bigcup_{p \in X-A} U_p$ is open! ✓

($\Leftarrow$) if $X-A$ is open, then for any $p \notin A$ we have

$$p \in X-A \quad \text{and} \quad ((X-A)-\{p\}) \cap A = \emptyset$$

so p is not a limit pt of A

i.e. A contains all its limit pts so A is closed ✓

3) if A, B are closed, then $(X-A)$, $(X-B)$ are open

so $X-(A \cup B) = (X-A) \cap (X-B)$ is open

de Morgan's Law

$\therefore$ $A \cup B$ is closed

4) almost same as proof of 3)

<u>exercise</u>: check 1)

natural numbers

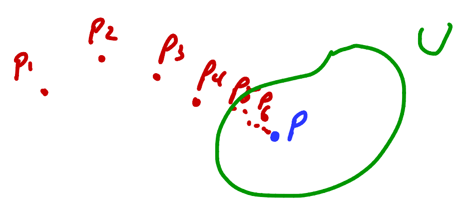
a <u>sequence</u> in X is a function $p: \mathbb{N} \longrightarrow X$

we denote $p(n)$ by p_n and the sequence by $\{p_n\}$

a sequence $\{p_n\}$ <u>converges to p</u> if for every open set U containing p there is some number N such that

$$p_n \in U \quad \text{for all } n \geq N$$

we denote this $p_n \longrightarrow p$



exercise: Show if (X,d) is a metric space then

$\{p_n\}$ converges to p (in metric topology)

$\Longleftrightarrow$

$\forall \varepsilon>0\ \exists N$ such that $d(p_n,p)<\varepsilon\ \forall n\geq N$

so convergence in metric spaces is just like from analysis class

lemma 3:

$\boxed{\text{let } A \text{ be a set in a topological space } (X,\mathcal{T}). \text{ If } \exists \text{ a sequence } \{p_n\} \text{ in } A \text{ and } p_n\to p, \text{ then } p\in\bar{A}}$

Proof: if $p\in A$, then $p\in\bar{A}$ so done

if $p\notin A$, then for any open set U containing p, since $p_n\to p$

$\exists N$ s.t. $\forall n\geq N$, $p_n\in U$

note $p_n\in A$, $p\notin A$, so $p_n\neq p$

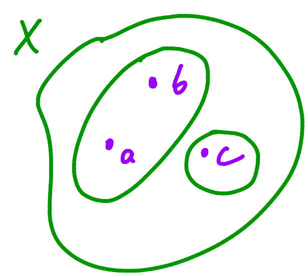
$\therefore\ (U-\{p\})\cap A\neq\varnothing$ (contains p_n, $\forall n\geq N$)

thus p is a limit pt. of A and so $p\in\bar{A}$ ▦

Remark: Sequences can behave strangely in a general topological space

for example: $X=\{a,b,c\}$

$\mathcal{T}=\{\varnothing, X, \{a,b\}, \{c\}\}$



note: the sequence

$p_n=a\ \forall n$

converges to a and to b!

What went wrong?

answer: not enough open sets to "distinguish" a and b

also recall from analysis you expect that if p is a limit point of A then $\exists$ a sequence $\{p_n\}$ in A such that $p_n \to p$

but in a general topological space that is not true!

How can we fix these problems?

a topological space $(X, \mathcal{T})$ is called Hausdorff if for every pair of distinct points $x, y \in X$ there are disjoint open sets U and V such that $x \in U$ and $y \in V$

we call a collection $\mathcal{N}$ of open sets in X containing $p \in X$ a neighborhood basis for p if for every open set U containing p, there is some set $V \in \mathcal{N}$ such that $p \in V \subset U$

we call $(X, \mathcal{T})$ 1^{st} countable if every point $p \in X$ has a countable neighborhood basis

lemma 4:

If $(X, \mathcal{T})$ is a Hausdorff topological space and $\{p_n\}$ is a sequence in X that converges to p and to q, then $p = q$.

Proof: If $p \neq q$, then $\exists$ disjoint open sets U and V such that $p \in U$ and $q \in V$

since $p_n \to p$, $\exists N$ such that $p_n \in U$ and $p_n \in V$, $\forall n \geq N$

$\therefore U \cap V \neq \emptyset$

this contradicts disjointness of U and V, so we must have $p = q$ ∎

lemma 5:

let $(X, \mathcal{T})$ be a 1^{st} countable topological space

If p is a limit point of A, then $\exists$ a sequence $\{p_n\}$ in A such that $p_n \to p$

Proof: let $\{V_i\}_{i=1}^{\infty}$ be a neighborhood basis for p

Set $U_1 = V_1$

$U_2 = U_1 \cap V_2 = V_1 \cap V_2$

$\vdots$

$U_n = U_{n-1} \cap V_n = V_1 \cap V_2 \cap \dots \cap V_n$

$\vdots$

note: $U_1 \supset U_2 \supset \dots \supset U_n \supset \dots$

exercise: Show $\{U_i\}_{i=1}^{n}$ is also a neighborhood basis for p (called nested neighborhood basis)

now if $p \in A$, then take $p_n = p$ for all n, and we see $p_n \to p$

if $p \notin A$, then note $(U_n - \{p\}) \cap A \neq \emptyset$ $\forall n$ since p a limit pt of A

so pick $p_n \in (U_n \cap A)$

note $\{p_n\}$ is a sequence in A

Claim: $p_n \to p$

indeed, if U is any open set containing p

then since $\{U_n\}$ a nbhd basis for p

$\exists$ some N s.t. $U_N \subset U$ $\therefore U_n \subset U$ $\forall n \geq N$ (since nested)

$\therefore p_n \in U_n \subset U$ $\forall n \geq N$, that is $p_n \to p$ ∎

Th^m 6:

1) metric spaces are Hausdorff and 1^{st} countable
2) subspaces of Hausdorff spaces are Hausdorff
   " " 1^{st} countable " " 1^{st} countable
3) products of Hausdorff spaces are Hausdorff
   " " 1^{st} countable " " 1^{st} countable

Proof: 1) Hausdorff: if $x \neq y$ in a metric space (X, d), then $c = d(x,y) > 0$

note $B_{c/2}(x) \cap B_{c/2}(y) = \emptyset$ ✓

<u>1st Countable:</u> given $x \in X$, then $\{B_{1/n}(x)\}_{n=1}^{\infty}$ can easily be checked to be a nbhd basis ✓

<u>exercise:</u> Check 2) and 3)

C. Continuous Functions

let $(X,\mathcal{T})$ and $(Y,\mathcal{T}')$ be two topological spaces

a function
$$f: X \to Y$$
is <u>continuous</u> if $f^{-1}(U)$ is an open set if X for all open sets U in Y

this means $\{x \in X : f(x) \in U\}$

<u>exercise:</u> You can easily check

$f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous
(using standard metric topologies)
$$\Longleftrightarrow$$
$\forall \varepsilon > 0$ and $x \in \mathbb{R}^n$, $\exists \delta > 0$ such that
$$d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$$

so continuity generalizes what you know from analysis

<u>Thm 7:</u>

for a function $f: X \to Y$ the following are equivalent
1) f is continuous
2) $f^{-1}(C)$ is closed in X for all closed C in Y
3) for any $A \subset X$, $f(\overline{A}) \subseteq \overline{f(A)}$

<u>Proof:</u> <u>1) ⇒ 2):</u>

We first note that for any $A \subset Y$: $f^{-1}(Y - A) = X - f^{-1}(A)$

<u>indeed:</u> $\subseteq$: $x \in f^{-1}(Y-A) \Rightarrow f(x) \in Y - A$, so $f(x) \notin A$
$\therefore x \notin f^{-1}(A)$ and so $x \in X - f^{-1}(A)$

$\supseteq$: $x \in X - f^{-1}(A) \Rightarrow x \notin f^{-1}(A)$ so $f(x) \notin A$
$\therefore f(x) \in Y - A$, thus $x \in f^{-1}(Y-A)$ ✓

now if f is continuous and $C \subset Y$ is closed

then $Y - C$ is open and thus $f^{-1}(Y-C) = X - f^{-1}(C)$ is open

hence $f^{-1}(C)$ closed $\therefore$ 2) is true ✓

2) ⇒ 1) is same argument ✓

3) ⇒ 2): let C be closed in Y

Set $A = f^{-1}(C)$

if $x \in \bar{A}$, then $f(x) \in f(\bar{A}) \overset{\text{by 3)}}{\subseteq} \overline{f(A)} \overset{\text{def}^{\text{n}}\text{ of } A}{=} \overline{f(f^{-1}(C))} \subseteq \bar{C} = C$

so $x \in f^{-1}(C) = A$ and $A = \bar{A}$ is closed

$f(f^{-1}(C)) \subseteq C$

$x \in f(f^{-1}(C))$

$\Rightarrow \exists y \in f^{-1}(C)$,

s.t. $f(y) = x$

so $x = f(y) \in C$

1) ⇒ 3): given $p \in \bar{A}$

if $p \in A$, then $f(p) \in f(A) \subset \overline{f(A)}$ ✓

if $p \notin A$, then p is a limit point of A

if $f(p) \in f(A)$ then done so assume $f(p) \notin f(A)$

Claim: $f(p)$ is a limit point of $f(A)$

($\therefore f(p) \in \overline{f(A)}$ and done)

to see this suppose $f(p)$ not a limit point of $f(A)$

thus $\exists$ an open set U in Y s.t. $f(p) \in U$

and $U \cap f(A) = \emptyset$

we know $f^{-1}(U)$ is open in X (since f cont.)

and $p \in f^{-1}(U)$

also $f^{-1}(U) \cap A \overset{\text{hopefully obvious}}{\subseteq} f^{-1}(U) \cap f^{-1}(f(A))$

$\overset{*}{=} f^{-1}(U \cap f(A)) = f^{-1}(\emptyset) = \emptyset$

so p is not a limit point of A ⊗ choice of p

$\therefore f(p)$ is a limit pt of $f(A)$ ▦

* $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

$\subseteq$: obvious since $A \cap B \subset A$ and $A \cap B \subset B$

$\supseteq$: if $x \in f^{-1}(A) \cap f^{-1}(B)$

then $f(x) \in A$ and B

$\therefore f(x) \in A \cap B$

and $x \in f^{-1}(A \cap B)$

Th^m 8:

If X is 1^{st} countable
Then $f: X \to Y$ is continuous
$\Leftrightarrow$
for each sequence $p_n \to p$ in X
we have $f(p_n) \to f(p)$ in Y

Proof: $(\Rightarrow)$ let $p_n \to p$ in X

let U be an open set in Y such that $f(p) \in U$

then $f^{-1}(U)$ open in X and $p \in f^{-1}(U)$

so $\exists N$ such that $n \geq N \Rightarrow p_n \in f^{-1}(U)$

$\therefore$ $f(p_n) \in f(f^{-1}(U)) \subset U \quad \forall n \geq N$

i.e. $f(p_n) \to f(p)$

note: this implication does not need 1^{st} count.

$(\Leftarrow)$ let A be a set in X

we show $f(\bar{A}) \subset \overline{f(A)}$ then done by Th^m 7

so take $p \in \bar{A}$

if $p \in A$, then $f(p) \in f(A) \subset \overline{f(A)}$ ✓

if $p \notin A$, then p a limit pt of A

so by lemma 5 $\exists$ a sequence $\{p_n\}$ in A
s.t. $p_n \to p$

now $f(p_n) \to f(p)$ in Y and $\{f(p_n)\}$ a sequence in $f(A)$

$\therefore$ lemma 3 $\Rightarrow$ $f(p) \in \overline{f(A)}$

so $f(\bar{A}) \subset \overline{f(A)}$ ▦

examples of continuous maps:

1) if $y_0 \in Y$ a point, then the constant map

$$f: X \to Y : x \mapsto y_0$$

is continuous, since for any open set $U \subset Y$

$$f^{-1}(U) = \begin{cases} \emptyset & p \notin U \\ X & p \in U \end{cases} \quad \text{is open in } X$$

2) if A a subspace of X, then the inclusion map

$$i : A \longrightarrow X : x \longmapsto x$$

is continuous, since for any open set $U \subset X$

$i^{-1}(U) = U \cap A$ is open in A

3) if $f : X \to Y$ is continuous and $A \subset X$ has the subspace topology, then the restriction

$$f|_A : A \longrightarrow X$$

is continuous, since for any open $U \subset Y$

$(f|_A)^{-1}(U) = f^{-1}(U) \cap A$ is open in A

4) projection maps are continuous

$$\pi_1 : X \times Y \longrightarrow X : (x, y) \longmapsto x$$

($X \times Y$ has the product topology)

since for any open set U in X

$\pi_1^{-1}(U) = U \times Y$ is open in $X \times Y$

similarly for $\pi_2 : X \times Y \to Y : (x, y) \longmapsto y$

5) compositions of continuous maps are continuous

$$f : X \to Y, \quad g : Y \to Z$$

$$g \circ f : X \longrightarrow Z : x \longmapsto g(f(x))$$

since if U is open in Z, then $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$

and $g^{-1}(U)$ open in Y so $f^{-1}(g^{-1}(U))$ open in X

6) $F : Z \to X \times Y : z \longmapsto (f(z), g(z))$ is continuous

$$\Longleftrightarrow$$

$f : Z \to X$ and $g : Z \to Y$ are continuous

indeed: ($\Rightarrow$) follows since $f = \pi_1 \circ F$ and $g = \pi_2 \circ F$ (by 4), 5))

($\Leftarrow$) exercise

Th^{m} 9:

let $(X, \mathcal{T})$ be a topological space and $X = A \cup B$ with A and B closed sets in X

If 1) $f: A \to Y$ and $g: B \to Y$ are continuous and

2) $f(x) = g(x)$ for all $x \in A \cap B$

Then there is a unique continuous map

$$h: X \to Y$$

such that $\forall x \in A$, $h(x) = f(x)$ and $\forall x \in B$, $h(x) = g(x)$

Proof:

define $h: X \to Y: x \longmapsto \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$

by 2), h is clearly well-defined

we show $h^{-1}(C)$ closed for any closed C in Y (then h continuous by Th^{m} 7)

Claim: $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

Pf: $(\subseteq)$ $x \in h^{-1}(C) \subset X$

so $x \in A$ or $x \in B$, say $x \in A$ (other case similar)

so $h(x) = f(x) \therefore f(x) \in C$ and $x \in f^{-1}(C) \subset f^{-1}(C) \cup g^{-1}(C)$ ✓

$(\supseteq)$ $x \in f^{-1}(C) \cup g^{-1}(C)$

suppose $x \in f^{-1}(C)$ (other case similar)

so $x \in A$ and $h(x) = f(x) \in C$ so $x \in h^{-1}(C)$ ✓

f, g continuous $\Rightarrow$ $f^{-1}(C)$ closed in A and $g^{-1}(C)$ closed in B

exercise: Since A and B are closed in X, show $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X

$\therefore h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ is closed in X (by lemma 2) ▦

a function $f: X \to Y$ is a <u>homeomorphism</u> if f is a continuous bijection and the inverse function $f^{-1}: Y \to X$ is also continuous

> This is the natural equivalence between topological spaces

we say X and Y are <u>homeomorphic</u> if there is a homeomorphism from one to the other

<u>note</u>: all questions about continuity, convergence, and the like are exactly the same in homeomorphic spaces

so from the perspective of topology, you should think of homeomorphic spaces as the same

<u>examples</u>:

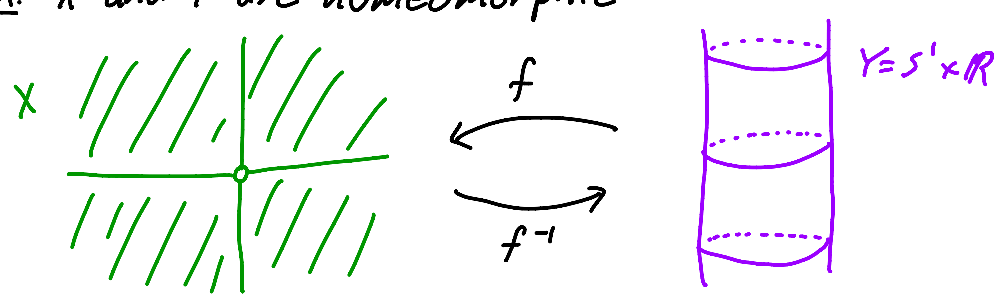
1) let $X = \mathbb{R}^2 - \{(0,0)\}$ with the subspace topology

$Y = S^1 \times \mathbb{R}$ with the product topology

(S^1 gets subspace topology from $\mathbb{R}^2$

where $S^1 = \{(a,b) : a^2 + b^2 = 1\}$

<u>Claim</u>: X and Y are homeomorphic



so while X and Y "look" different they are really the same! (topologically)

$$f((a,b),z) = (e^z a, e^z b)$$

$$g(x,y) = \left(\frac{(x,y)}{\sqrt{x^2+y^2}}, \ln\sqrt{x^2+y^2} \right)$$

on unit circle

well-defined since $x^2+y^2>0$

note: $f \circ f^{-1}(x,y) = \left(e^{\ln\sqrt{x^2+y^2}} \frac{x}{\sqrt{x^2+y^2}}, e^{\ln\sqrt{x^2+y^2}} \frac{y}{\sqrt{x^2+y^2}} \right)$

$$= (x,y)$$

$$f^{-1} \circ f((a,b),z) = \left(\frac{(e^z a, e^z b)}{\sqrt{e^{2z}\underbrace{(a^2+b^2)}_{=1}}}, \ln\sqrt{e^{2z}\underbrace{(a^2+b^2)}_{=1}} \right)$$

$$= ((a,b),z)$$

so f is a bijection with inverse f^{-1}

from calculus we know $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} : (x,y,z) \mapsto xe^z$

is continuous, so restricting to $S^1 \times \mathbb{R}$

also continuous

similarly for $(x,y,z) \mapsto ye^z$

so f is continuous since its component functions are.

you can similarly use calculus to see f^{-1} is continuous

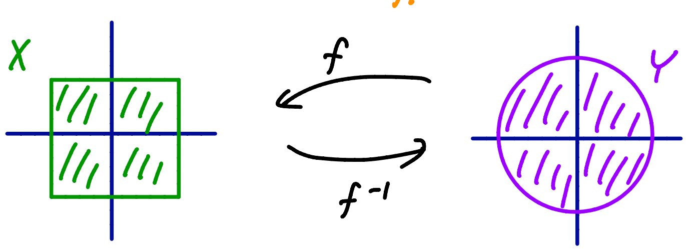
so f is a homeomorphism!

2) let $X = [-1,1] \times [-1,1] = \{(x,y) \in \mathbb{R}^2 : |x| \le 1, |y| \le 1\}$

$Y = D^2 = \{(x,y) \mid x^2+y^2 \le 1\}$

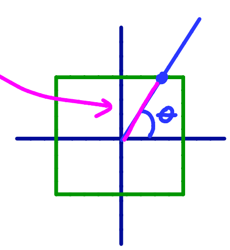
<u>Claim</u>: X and Y are homeomorphic

(so topology doesn't "see" corners)



there is a continuous function $g : S^1 \to (0,\infty)$

such that $g(\theta)$ gives length



indeed $g(\theta) = \begin{cases} |\cos\theta|^{-1} & \theta \in [-\frac{\pi}{4}, \frac{\pi}{4}] \cup [\frac{3\pi}{4}, \frac{5\pi}{4}] \\ |\sin\theta|^{-1} & \theta \in [\frac{\pi}{4}, \frac{3\pi}{4}] \cup [\frac{5\pi}{4}, \frac{7\pi}{4}] \end{cases}$

<u>exercise:</u> g is continuous (use Th$^{\underline{m}}$ 9)

now $f(r,\theta) = (g(\theta)r, \theta)$ (polar coordinates)

$f^{-1}(r,\theta) = (\frac{1}{g(\theta)} r, \theta)$

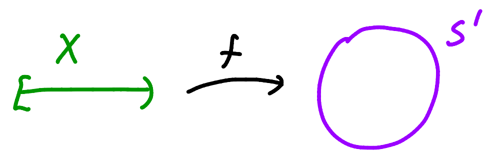
clearly f a bijection with inverse f^{-1}

and f and f^{-1} are continuous (why?)

<u>Remark:</u> It is very important in the definition of homeomorphism that f^{-1} is continuous

<u>example:</u> $X = [0,1)$

$Y = S^1$



$f : X \to Y : t \longmapsto (\cos 2\pi t, \sin 2\pi t)$

if we think of f as a map $X \to \mathbb{R}^2$ it is easy to see from calculus that f is continuous

this implies $f : X \to Y$ is continuous

(since U open in Y means $\exists V$ open in $\mathbb{R}^2$ such that $U = S^1 \cap V$

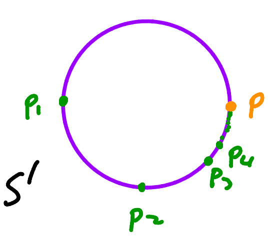
and $f^{-1}(U) = f^{-1}(S^1 \cap V) = f^{-1}(V)$ open in X)

so f is a continuous bijection, but we don't want to think of the interval and S^1 as the same!

luckily they arn't because

<u>Claim:</u> f^{-1} is not continuous

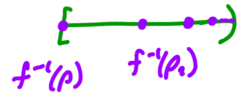
indeed let $p_n = f(1 - \frac{1}{n})$



this is a sequence $\{p_n\}$ in S^1

and $p_n \to p = (1,0)$ in S^1

but $f^{-1}(p) = 0$



so $f^{-1}(p_n)$ does not converge to $f^{-1}(p)$

$\therefore$ f^{-1} is <u>not</u> continuous ✓

an injective continuous map $f: X \to Y$ is called an <u>embedding</u> if $f: X \to f(X)$ is a homeomorphism where $f(X) \subset Y$ has the subspace topology

so if we have an embedding $X \to Y$ then we may think of X as a subspace of Y

<u>example:</u> if $A \subset X$ is a subspace, then the inclusion map $\imath: A \to X$ is an embedding

knots give interesting embeddings of S^1 in $\mathbb{R}^3$